Managing Smile Risk

Hagan et al.'s Stochastic - $\alpha\beta\rho$ model

Cheng Shantian

Outline

Black-Scholes-Merton Model Assumptions

Stock price F is modeled by the geometric Brownian motion

$$\mathrm{d}F_t = \alpha F_t \,\mathrm{d}t + \sigma F_t \,\mathrm{d}W_t.$$

- Constant drift α and volatility σ;
- Under the risk-neutral measure, α is risk-free rate r.

Black-Scholes-Merton Formula

European call option pays $(F(T) - K)^{+}$ at time T.

Black-Scholes-Merton (BSM) model on European call option price is given as

$$BSM(\tau, S; K, r, \sigma) = FN\left(d_{+}(\tau, F)\right) - Ke^{-r\tau}N\left(d_{-}(\tau, F)\right),$$

where

$$d_{\pm}(\tau, F) = \frac{\log\frac{F}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}.$$

Implied Volatility

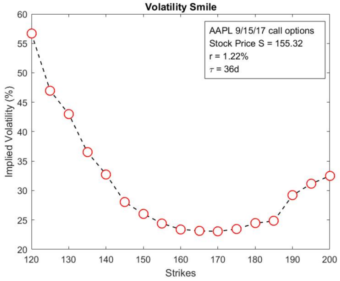
Implied volatility is the parameter σ that makes the BSM option price agree with the market price

$$BSM(\tau, S; K, r, \sigma_{\text{imp}}) = V_{\text{call}}^{\text{market}}.$$

There is a one-to-one, monotonic correspondence between prices and implied volatility.

Volatility Smile

In reality, the implied volatility calculated from different options (across strikes, maturities, dates) are usually different.



Data retrieved from Bloomberg on 20170811.

We need models that are consistent in different options on the same underlying, and moreover, that can capture the dynamics behaviors of volatility smiles.

Outline

Stochastic-$\alpha\beta\rho$ Model

Stochastic-$\alpha\beta\rho$ Model (SABR)

$$
\begin{aligned}
\mathrm{d}F_t &= \alpha_t F_t^{\beta} \mathrm{d}W_t^1, \qquad & F_0 &= f \\
\mathrm{d}\alpha_t &= \nu \alpha_t \mathrm{d}W_t^2, \qquad & \alpha_0 &= \alpha \\
\mathrm{d}W_t^1 \mathrm{d}W_t^2 &= \rho \mathrm{d}t
\end{aligned}
$$

The parameters are

- α the initial volatility
- ν the volatility of volatility
- β the exponent for the underlying price
- ρ the correlation between the two Brownian motions

SABR Implied Volatility - General

The implied volatility $\sigma_B(K, f)$ is given by

$$\sigma_B(K,f) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\log^2 f/K + \frac{(1-\beta)^4}{1920}\log^4 f/K\right\}} \cdot \left(\frac{z}{x(z)}\right) \cdot$$
$$\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]\tau + \cdots\right\}$$

where z is defined by

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\log f/K$$

and $x(z)$ is given by

$$x(z) = \log\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right).$$

SABR Implied Volatility - fine-tuned in Obłój 2008

Given parameters α, β, ρ, ν, the SABR implied volatility is given by

$$\sigma_B = I(K, f, \tau) = I^0(K, f)\left[1 + I^1(K, f)\tau\right] + O(\tau^2),$$

where

$$I^1(K, f) = \frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2,$$

and $I^0(K, f)$ has two versions shown in the following slide (Obłój recommended the second version)

$I^0(K, f)$

	Hagan et al. 2002 VS Berestycki et al. 2004	Notation
$I^0(K,K)$	$\alpha K^{\beta-1} = \alpha K^{\beta-1}$	
$I^0(K,f)\|_{\nu=1}$	$\frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}} = \frac{x\alpha(1-\beta)}{s^{1-\beta}-K^{1-\beta}}$	$x = \log \frac{f}{K}$
$I^0(K,f)\|_{\beta=1}$	$\nu x / \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right) = \nu x / \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$	$x = \log \frac{f}{K}$ $z = \frac{\nu x}{\alpha}$
$I^0(K,f)\|_{\beta<1}$	$\nu x \frac{\zeta}{z} / \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right) \neq \nu x / \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$	$x = \log \frac{f}{K}$ $z = \frac{\nu}{\alpha}\frac{s^{1-\beta}-K^{1-\beta}}{1-\beta}$ $\zeta = \frac{\nu}{\alpha}\frac{s-K}{(sK)^{\beta/2}}$

SABR Implied Volatility - ATM

For at-the-money options (K=f) the formula reduces to

$$\sigma_{\text{ATM}} = \frac{\alpha \left\{ 1 + \left[\frac{(1-\beta)^2}{24} \frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4} \frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2 \right] \tau + \cdots \right\}}{f^{1-\beta}}$$

Hence, given σ_{ATM}, approximately, α is the smallest positive root of the cubic (quadratic for $\beta = 1$) equation

$$\left[\frac{(1-\beta)^2 \tau}{24 f^{2-2\beta}} \right] \alpha^3 + \left[\frac{\rho\beta\nu\tau}{4 f^{1-\beta}} \right] \alpha^2 + \left[1 + \frac{2-3\rho^2}{24} \nu^2 \tau \right] \alpha - \sigma_{ATM} f^{1-\beta} = 0$$

Parameter Estimation

With any specific value of β, market smile can be fit equally well.

- Stochastic lognormal: $\beta = 1$; normal: $\beta = 0$; CIR: $\beta = \frac{1}{2}$.

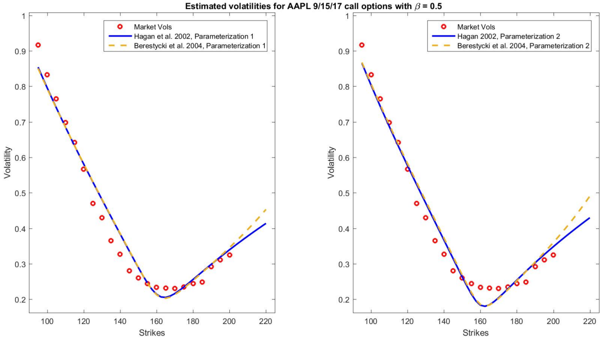
First Parameterization:

$$(\hat{\alpha}, \hat{\rho}, \hat{\nu}) = \text{argmin}_{\alpha,\rho,\nu} \sum_i \left(\sigma_i^{\text{mkt}} - \sigma_B(f_i, K_i; \alpha, \rho, \nu)\right)^2$$

Second Parameterization:

- $\alpha = \alpha(\rho, \nu; \sigma_{ATM}, \tau)$ based on the algebraic formula of σ_{ATM};
- $$(\hat{\rho}, \hat{\nu}) = \text{argmin}_{\rho,\nu} \sum_i \left(\sigma_i^{\text{mkt}} - \sigma_B\left(f_i, K_i; \alpha(\rho, \nu; \sigma_{ATM}, \tau), \rho, \nu\right)\right)^2$$

Fit Market Smile

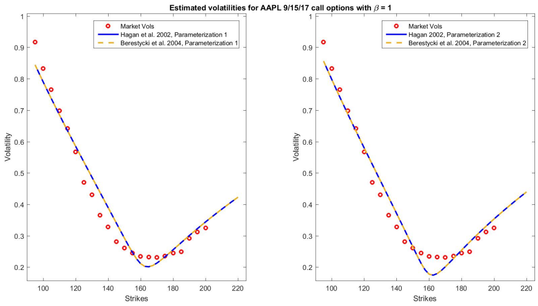


	α	ρ	ν	SSE
Hagan et al. 2002, Parameterization 1	2.8949	−0.5900	3.7296	0.0272
Berestycki et al. 2004, Parameterization 1	2.9044	−0.6062	3.7328	0.0288
Hagan et al. 2002, Parameterization 2	2.4893	−0.5823	4.1137	0.0226
Berestycki et al. 2004, Parameterization 2	2.5542	−0.6061	4.0636	0.0254

Table: Parameters in different parameterization methods with $\beta = 0.5$

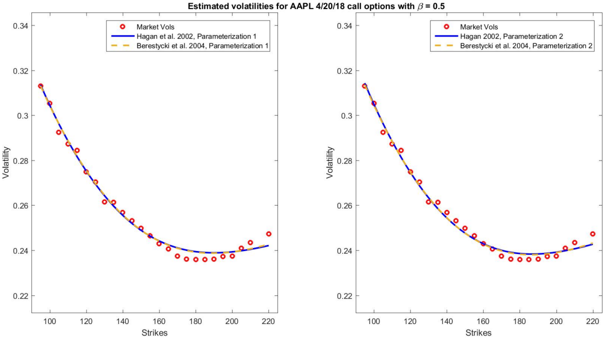
Initial volatility $\alpha > 2$?

β and α control the scale level of the smile (will be illustrated later). As β is fixed, α has to be large to in calibration to fit the scale of smile. $\beta = 1$ is recommended for options on stock.



	α	ρ	ν	SSE
Hagan et al. 2002, Parameterization 1	0.2354	−0.6418	3.9647	0.0330
Berestycki et al. 2004, Parameterization 1	0.2354	−0.6418	3.9647	0.0330
Hagan et al. 2002, Parameterization 2	0.1997	−0.6323	4.3819	0.0279
Berestycki et al. 2004, Parameterization 2	0.1997	−0.6323	4.3819	0.0279

Fine-Tuned Smile Better for Large Maturities

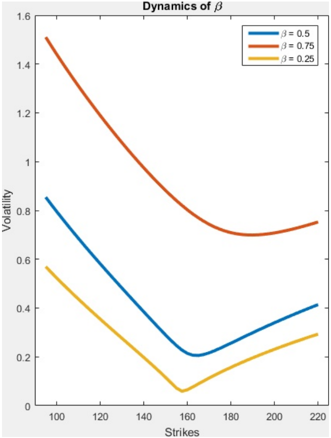


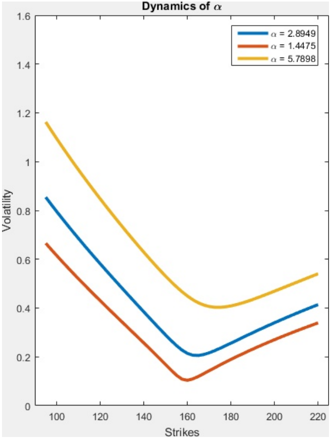
	α	ρ	ν	SSE
Hagan et al. 2002, Parameterization 1	3.0244	-0.0448	0.4993	1.4772e-4
Berestycki et al. 2004, Parameterization 1	3.0244	-0.0517	0.5005	1.3323e-4
Hagan et al. 2002, Parameterization 2	3.0048	-0.0414	0.5201	1.2998e-4
Berestycki et al. 2004, Parameterization 2	3.0039	-0.0483	0.5221	1.1399e-4

Table: Parameters in different parameterization methods with $\beta = 0.5$

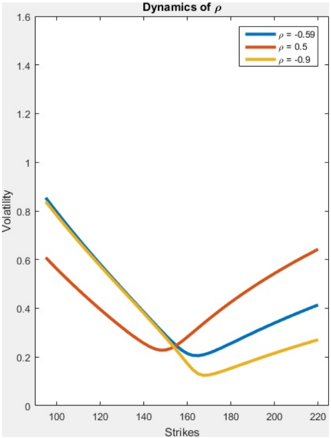
Parameter Dynamics

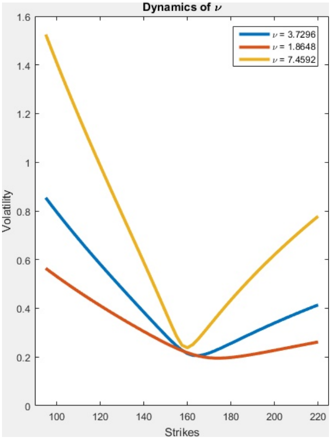
With base estimation $(\beta, \alpha, \rho, \nu) = (0.5, 2.8949, -0.59, 3.7296)$, let only one parameter vary and the effect of each parameter on the shape of the curve is shown as follows:





Parameter Dynamics

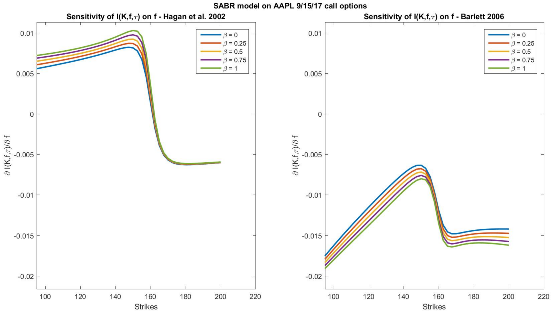




Greeks

$\frac{\partial \sigma_B(K,f;\alpha(f),\beta,\rho,\nu)}{\partial f}$: Sensitivity of implied volatility to underlying price change

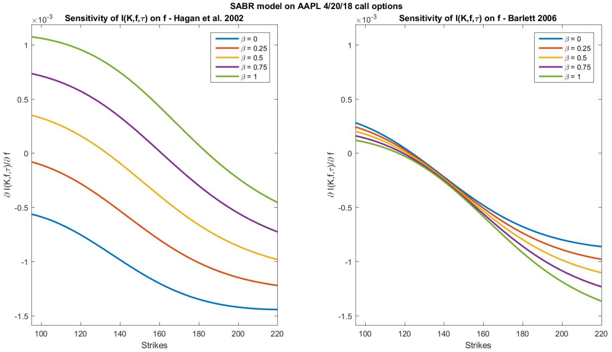
- Hagan et al. 2002: $\frac{\partial \alpha}{\partial f} = 0$ in first parameterization
- Barlett 2006: $\frac{\partial \alpha}{\partial f} = \frac{\rho \nu}{f^\beta}$



Note that in this example, Hagan et al's smile moves in the same direction as the underlying, while Barlett's smile is not consistent with the market behavior.

Barlett's Smile Better for Large Maturities

In my experiment, Barlett's smile moves in the same directions as the underlying for large maturities, and is robust with respect to β.



Risk decomposition [Hagan and Lesniewski 2017]

Decompose Brownian motion based on ρ, we get

$$\mathrm{d}\alpha = \frac{\rho\nu}{f^{\beta}}\mathrm{d}F + \nu\alpha\sqrt{1-\rho^2}\mathrm{d}\alpha^{\perp}$$

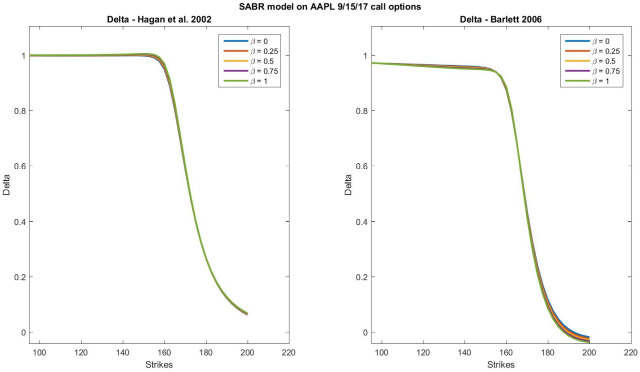
The risk decomposition of an option in terms of independent risk factors $\mathrm{d}F$, $\mathrm{d}\alpha^{\perp}$, time decay and second order Greeks is as follows:

$$\begin{aligned}\mathrm{d}V_{\text{call}} = &\left\{-\text{Theta}_t + \frac{1}{2}\alpha_t^2\left(F^{2\beta}\text{Gamma}_t + 2F^{\beta}\text{Vanna}_t + \nu^2\text{Volga}_t\right)\right\}\mathrm{d}t \\ &+ \text{Delta}_t^{\text{mod}}\mathrm{d}F_t + \text{Vega}_t\mathrm{d}\alpha_t^{\perp}\end{aligned}$$

Greeks - Delta

Delta$^{\text{mod}}$ (Hagan and Lesniewski ,2017): Sensitivity to underlying price change

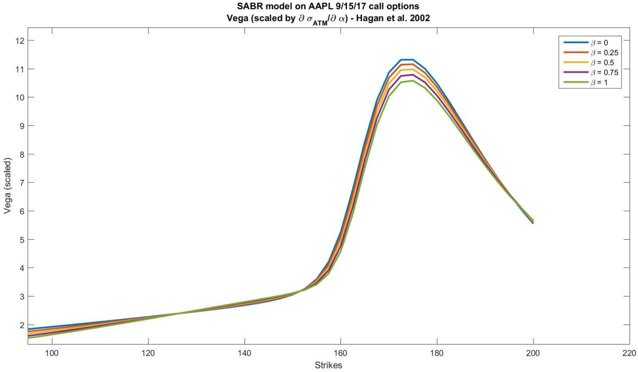
$$\frac{\partial V_{\text{call}}}{\partial f} = \frac{\partial \text{BSM}}{\partial f} + \frac{\partial \text{BSM}}{\partial \sigma_B} \cdot \left(\frac{\partial \sigma_B(K, f; \alpha, \beta, \rho, \nu)}{\partial f} + \frac{\partial \sigma_B}{\partial \alpha} \cdot \frac{\rho \nu}{f^{\beta}} \right)$$



Greeks - Vega

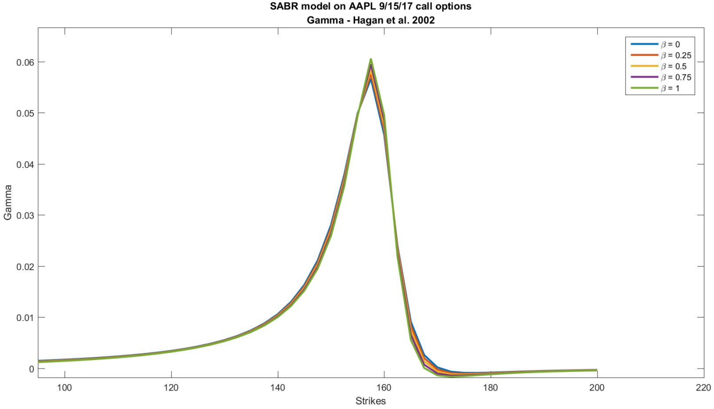
Vega (scaled): Sensitivity to ATM volatility change

$$\frac{\partial V_{\text{call}}}{\partial \alpha} = \frac{\partial \text{BSM}}{\partial \sigma_B} \cdot \frac{\frac{\partial \sigma_B(K, f; \alpha, \beta, \rho, \nu)}{\partial \alpha}}{\frac{\partial \sigma_{ATM}(K, f; \alpha, \beta, \rho, \nu)}{\partial \alpha}}$$



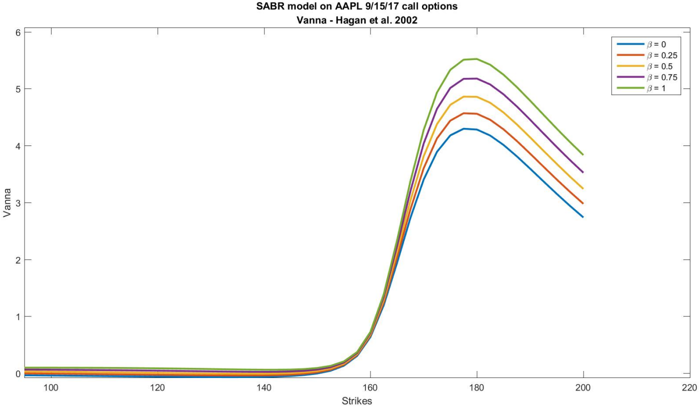
Greeks - Gamma

Gamma: Sensitivity to Delta^{mod} change



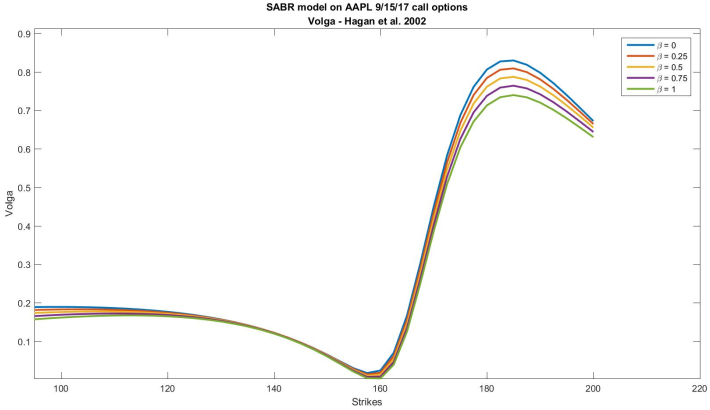
Greeks - Vanna

Vanna: Sensitivity to correlation change



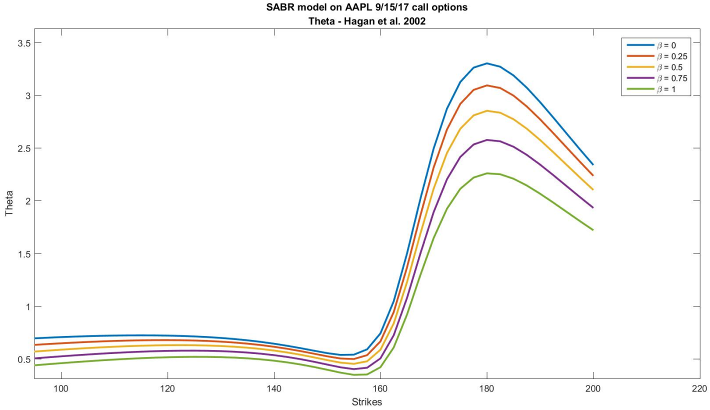
Greeks - Volga

Volga: Sensitivity to "volatility of volatility" change



Greeks - Theta

Theta: Sensitivity to time decay



Outline

Further Challenges

1. Hagan et al's formula may imply inaccurate parameters for large maturities.
   - Obłój's refinements may improve.
2. Hagan et al's SABR ignore the mean-reversion property of volatility for large maturities.
   - Heston model, Karasinski and Sepp's Beta model
   - λ-SABR models using numerical techniques, geometric approach, Malliavin calculus
3. Parameters of Hagan et al's SABR are not time-dependent.
   - Dynamic SABR Model using numerical techniques

Reference

- Bartlett, B. (2006). ***Hedging Under SABR Model***, Wilmott Magazine, July 2006, pp. 2-4.
- Berestycki, H., Busca, J. and Florent. I. (2004) ***Computing the Implied Volatility in Stochastic Volatility Models***, Comm. Pure Appl. Math., 57(10), pp. 1352-1373.
- Hagan, P., Kumar, D., Lesniewski, A., and Woodward, D. (2002). ***Managing Smile Risk***, Wilmott Magazine, September 2002, pp. 84-108.
- Hagan, P. and Lesniewski, A. (2017). ***Bartlett's delta in the SABR model***, preprint.
- Obłój, J. (2008). ***Fine-Tune Your Smile: Correction to Hagan et al***, Wilmott Magazine, May 2008, pp. 102-104.

THANK YOU

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