Tate's Thesis : the function field case

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1 Basic Propositions of Function Field

We use the same notation as the number field case.

A global field is a finite extension of $\mathbb{Q}$ or $\mathbf{F}_p(t)$.

Firstly, we recall a basic proposition of function field $\mathbf{F}_p(t)$:

Proposition 1.1. Let $K = \mathbf{F}_p(t)$ and $R = \mathbf{F}_p[t]$. Then every nontrivial place of K is given by either the "infinite place" $|\cdot|_\infty$ defined by

$$|f/g|_\infty = p^{deg(f)-deg(g)}$$

or by the finite place $|\cdot|_P$ corresponding to an irreducible polynomial $P(t) \in R$.

Proof. Similar with the proof of Ostrowski's Theorem, consider two cases:

(i) there exists a polynomial $P \in \mathbf{F}_p[t]$ such that $|P| < 1$;

(ii) for every irreducible polynomial $P \in \mathbf{F}_p[t]$ we have $|P| \geqslant 1$. □

For place P (for convenience here, P may be ∞), the valuation $|\cdot|_P$ can be embedded in the complete field $(\mathbf{F}_p(t))_P$. For K is a global field, the valuation of every finite place v of K is the unique extension of $|\cdot|_P$ where $v|P$ and

$$|\alpha|_v = |Norm_{K/\mathbf{F}_p(t)}\,\alpha|_P^{1/N} \qquad N = [K : \mathbf{F}_p(t)].$$

The completion of K respect to $|\cdot|_v$ is denoted by K_v.

The algebraic closure of $\mathbf{F}_p$ in K is a finite field, which is $\mathbf{F}_q$ where $q = p^m$ for some $m \geqslant 1$. That is to say, $\mathbf{F}_q$ is the largest finite subfield of K.

Proposition 1.2. Let K be a global field with char K=$p > 1$. Then

(i) For every $x \in K^*$ we have $|x|_{\mathbf{A}_K} = 1$;

(ii) The absolute value map $|\cdot|_{\mathbf{A}_K}$ has image of the form $q^{\mathbf{Z}}$.

Proof. (i) Consider an irreducible polynomial $P \in K$. This has nontrivial absolute value at only two places P and t^{-1};

(ii) Each component of $\mathbf{A}_K$ under the adelic absolute value is $q^{m\mathbf{Z}}$ (m=1 for infinite place), and hence the total image is $q^{\mathbf{Z}}$. □

Proposition 1.3. The completion of K

(i) in the place t^{-1}, is

$$K_\infty = \{\sum_{n=-\infty}^{r} a_n t^n : a_n \in \mathbf{F}_q\},$$

which has

$$\psi_\infty(x) = e^{2\pi i tr_{\mathbf{F}_q/\mathbf{F}_p}(a_1)/p} \in S^1$$

as a nontrivial character;

(ii) in the place v (v is any irreducible polynomial in K), is

$$K_v = \{\sum_{n=r}^{\infty} a_n v^n : a_n \in \textit{the residue field } k\},$$

which has

$$\psi_v(x) = e^{2\pi i tr_{k/\mathbf{F}_p}(a_{-1})/p} \in S^1$$

as a nontrivial character.

Proof. For $b \in K_v$, take an integer r satisfying $ord_v(b) \geqslant r$. Then $v^{-r}b$ is an element of O_v. So we can find an element $a_r \in k$ which coincides with the image of $v^{-r}b$ in k. We have $ord_v(b - a_r v^r) \geqslant r + 1$, then by induction. □

Note that if $s = t^{-1}$, so $\mathbf{F}_q(t) = \mathbf{F}_q(s)$, the valuation $|\cdot|_\infty$ is seen to be the type belonging to the irreducible polynomial $P(s) = s$. So every place of function field K is non-Archimedean, thus finite. The local theory of function field has no difference with that of number field. Particularly, the Local Multiplicity One Theorem and its proof holds true here (We only use the part of non-archimedean cases).

2 The Global Function Equation

We establish the normalization Haar measure d^*x on $\mathbf{A}_K^*$ given by the product measure whose factors are defined as follows:

$$\int_{O_v}^{*} d^*x_v = 1.$$

We recall the key formula in the global theory:

Theorem 2.1 (Poisson Summation Formula)**.** Let x be an idele of K and let f be an element of $S(\mathbf{A}_K)$. Then

$$\sum_{\gamma\in K} f(\gamma x) = |x|^{-1} \sum_{\gamma\in K} \hat{f}(\gamma x^{-1})$$

Then yields the main theorem here:

Theorem 2.2. Let K is a global field, the global Tate's integral

$$Z(s,\chi, f) = \int_{\mathbf{A}^*} f(x)\chi(x)|x|^s d^*x$$

has meromorphic continuation to all s, and satisfies the function equation

$$Z(s,\chi, f) = Z(1 - s,\chi^{-1}, \hat{f}).$$

The extended function $Z(s,\chi, f)$ is in fact holomorphic everywhere except when there exists a complex number λ such that $\chi(x) = |x|^\lambda$, in which case it has simple poles at $s = 1 - \lambda$ and $s = -\lambda$ with corresponding residues given by

$$\rho Vol(C_K^1)\hat{f}(0) \quad \textit{and} \quad -\rho Vol(C_K^1)f(0)$$

respectively, where

$$\rho = \begin{cases} 1 & \text{if } K \text{ is number field;} \\ \frac{1}{\log q} & \text{if } K \text{ is function field, char} K = p > 1, q \text{ is explained above.} \end{cases}$$

.

The key difference between number field and function field is

$$\mathbf{A}^*/\mathbf{A}_1^* = \begin{cases} \mathbb{R}_+^* & if\ K\ is\ a\ number\ field; \\ \mathbb{Z} & if\ K\ is\ a\ function\ field. \end{cases}$$

For K a function field, we have

$$Z(s, \chi, f) = \sum_{n \in \mathbb{Z}} Z_{t_n}(s, \chi, f)$$

where

$$Z_{t_n}(s, \chi, f) = \int_{\mathbf{A}_1^*} f(t_n x) \chi(t_n x) |t_n x|^s d^* x$$

with t_n is a set of representatives of $\mathbf{A}^*/\mathbf{A}_1^*$ and $|t_n| = q^n$. We can choose $t_{-n} = t_n^{-1}$, $t_0 = 1$.

Lemma 2.3. We have the relation

$$Z_{t_n}(s, \chi, f) = Z_{t_{-n}}(1 - s, \chi^{-1}, \hat{f}) + \hat{f}(0) \int_{C_K^1} \chi^{-1}(t_{-n}x) |t_{-n}x|^{1-s} d^* x - f(0) \int_{C_K^1} \chi(t_n x) |t_n x|^s d^* x$$

Proof. Applying the Poisson Summation Formula, we have

$$\begin{aligned} Z_{t_n}(s, \chi, f) + f(0) \int_{C_K^1} \chi(t_n x) |t_n x|^s d^* x &= \int_{C_K^1} \sum_{a \in K} f(a t_n x) \chi(t_n x) |t_n x|^s d^* x \\ &= \int_{C_K^1} |t_n x|^{-1} \sum_{a \in K} \hat{f}(a t_n^{-1} x^{-1}) \chi(t_n x) |t_n x|^s d^* x \\ &= \int_{C_K^1} |t_{-n} x| \sum_{a \in K} \hat{f}(a t_{-n} x) \chi(t_n x^{-1}) |t_n x^{-1}|^s d^* x \\ &= \int_{C_K^1} \sum_{a \in K} \hat{f}(a t_{-n} x) \chi^{-1}(t_{-n} x) |t_{-n} x|^{1-s} d^* x \\ &= Z_{t_{-n}}(1 - s, \chi^{-1}, \hat{f}) + \hat{f}(0) \int_{C_K^1} \chi^{-1}(t_{-n} x) |t_{-n} x|^{1-s} d^* x. \end{aligned}$$

□

Proof of the Theorem. From the lemma, we get that

$$Z(s, \chi, f) = Z_1(s, \chi, f) + \sum_{n \in \mathbb{Z}_+} Z_{t_n}(s, \chi, f) + \sum_{n \in \mathbb{Z}_+} Z_{t_n}(1 - s, \chi^{-1}, \hat{f}) + E'$$

where

$$E' = \sum_{n \in \mathbb{Z}_-} \left[\hat{f}(0) \chi^{-1}(t_{-n}) |t_{-n}|^{1-s} \int_{C_K^1} \chi^{-1}(x) d^* x - f(0) \chi(t_n) |t_n|^s \int_{C_K^1} \chi(x) d^* x \right]$$

And we can also write

$$Z_1(s,\chi,f) = \frac{1}{2}[Z_1(s,\chi,f) + Z_1(1-s,\chi^{-1},\hat{f})] + \frac{\hat{f}(0)}{2}\int_{C_K^1}\chi^{-1}(x)d^*x - \frac{f(0)}{2}\int_{C_K^1}\chi(x)d^*x$$

Putting $\epsilon_n = 1/2$ for $n = 0$ or $\epsilon_n = 1$ if $n > 0$, we then find that

$$Z(s,\chi,f) = \sum_{n\geqslant 0}\epsilon_n[Z_{t_n}(s,\chi,f) + Z_{t_n}(1-s,\chi^{-1},\hat{f})] + E$$

where

$$E = \begin{cases} 0 & if\ \chi \neq |\cdot|^{\lambda} \\ \sum_{n\geqslant 0}\epsilon_n Vol(C_K^1)[\hat{f}(0)q^{n(1-s-\lambda)} - f(0)q^{-n(s+\lambda)} & if\ \chi = |\cdot|^{\lambda}. \end{cases}$$

In the second case, we have

$$E = Vol(C_K^1)\Big[\frac{\hat{f}(0)}{1-q^{1-s-\lambda}} - \frac{f(0)}{1-q^{-s-\lambda}} - \frac{\hat{f}(0)-f(0)}{2}\Big].$$

The theorem now follows as in the number field case. □

3 Riamann-Roch Theorem

In the function field case, the Poisson Summation Formula can be interpreted to yield the Riamann-Roch Theorem of algebraic geometry. We shall show this after some preliminaries.

A divisor on K is a formal linear combination

$$D = \sum_v n_v v$$

where the sum runs over all places v of K and each coefficient n_v is an integer that is zero for almost all v. The divisors on K naturally form an additive proup, denoted $Div(K)$. The degree of a divisor $D = \sum_v n_v v$ is defined by

$$deg(D) = \sum_v n_v deg(v)$$

where $deg(v)$ is the degree of the residue field $\mathbf{F}_{q_v}$ over $\mathbf{F}_q$. Thus $q_v = q^{deg(v)}$.

Since $deg(D + D') = deg(D) + deg(D')$, we see that the degree map defines a homomorphism $deg : Div(K) \to \mathbf{Z}$, the kernel of which is denoted $Div^0(K)$, the group of divisors of degree 0.

Given any $f \in K^*$, we can associate a divisor, called a principal divisor, by setting

$$div(f) = \sum_v v(f)v$$

where $v(f)$ denotes the valuation of f at v. The quotient $Div(K)/div(K^*)$ is denoted $Pic(K)$ and called the Picard group of K.

Proposition 3.1. We have $div(K^*) \subseteq Div^0(K)$.

Proposition 3.2. We have the following exact sequence of groups:

$$1 \to \mathbf{F}_q^* \to K^* \to Div^0(K) \to Pic^0(K) \to 0$$

where $Pic^0(K)$ denotes the quotient $Div^0(K)/div(K^*)$.

$Div(K)$ is given the partial ordering defined by

$$D = \sum_v n_v v \geqslant D' = \sum_v n'_v v \qquad if \qquad n_v \geqslant n'_v.$$

With this, to each divisor D one may associate the following linear system of D:

$$L(D) = 0 \cup \{f \in K^* : div(f) \geqslant -D\}.$$

We have at once $L(D)$ is a vector space over $\mathbf{F}_q$, and write $l(D)$ for the dimension of the space. For example, $L(0) = \mathbf{F}_q$, $l(0) = 0$; if $deg(D) < 0$, $L(D) = 0$, $l(D) = 0$.

We may extend the divisor map from K to ideles $\mathbf{I}_K$:

$$div : \mathbf{I}_K \to Div(K)$$

$$(x_v) \mapsto \sum_v v(x_v)v$$

It is easy to see that this map is surjective.

Proposition 3.3. For any divisor D, $l(D)$ is finite.

Proof. Let $f = \otimes_v f_v \in S(\mathbf{A}_K)$ be defined by requiring that each component function f_v be the characteristic function on $\mathfrak{o}_v$. Given any divisor $D = \sum_v n_v v$, we may associate an idele $x(D)$ such that $v(x(D)_v) = n_v$ for all v. Then we have for all $\gamma \in K^*$ that

$$f(\gamma x(D)) = \begin{cases} 1 & if\ v(\gamma x(D)_v) \geqslant 0\ for\ \forall v (\Leftrightarrow \gamma \in L(D)) \\ 0 & otherwise \end{cases}$$

Note also that $f(0) = 1$.

Since $f \in S(\mathbf{A}_K)$, the sum $\sum_{\gamma \in K} f(\gamma x(D))$ converges. From our analysis above of $f(\gamma x(D))$ as a "counting function", the sum is exactly $Card(L(D))$ $(= q^{l(D)})$. Hence $l(D)$ is finite. □

Theorem 3.4 (Riemann-Roch, Geometric Form)**.** Let K is a global field with char$K = p > 1$. Then there exists an integer $g \geqslant 0$ (called the genus of K) and a divisor $\mathscr{K}$ of degree $2g-2$ (called the canonical divisor of K), such that

$$l(D) - l(\mathscr{K} - D) = deg(D) - g + 1$$

for every divisor D.

Proof. Let ψ be a nontrivial character of $\mathbf{A}_K/K$ and $P_v^{m_v}$ be the conductor of ψ_v for each place v. We get a divisor by setting

$$\mathscr{K} = -\sum_v m_v v.$$

Let $f = \otimes_v f_v \in S(\mathbf{A}_K)$ be defined above. We have seen that for each divisor $D = \sum_v n_v v$,

$$q^{l(D)} = \sum_{\gamma \in K} f(\gamma x(D)) = |x(D)|^{-1} \sum_{\gamma \in K} \hat{f}(\gamma x(D)^{-1})$$

with $x(D)$ defined as above. The second identity follows Poisson Summation Formula. Note that

$$|x(D)|^{-1} = \prod_v q_v^{n_v} = q^{(\sum_v n_v deg(v))} = q^{deg(D)}.$$

Recall that the Fourier transform is taken relative to the self-dual measure dx on $\mathbf{A}_K$ defined by ψ. In the local place, there exists $\delta_v \in K_v^*$ such that $\psi_{0,v}(x) = \psi_v(\delta_v^{-1}x)$ has conductor O. Then the conductor $P_v^{m_v}$ is equal to $\delta_v^{-1}O$.

$$\hat{f}_v(x) = \int_{K_v} f(y)\psi_v(xy)dy = \int_{O_v} \psi_{0,v}(\delta_v xy)|\delta_v|^{1/2}d^0y = |\delta_v|^{1/2}f(\delta_v x).$$

Then

$$\hat{f}_v = |\delta_v|^{1/2} \cdot Char(P_v^{m_v}).$$

Note that

$$|\delta_v|^{1/2} = q_v^{-(-m_v)} = q^{deg(v)m_v/2}.$$

So that $\prod_v |\delta_v|^{1/2} = q^{1-g}$. Thus we have for $\forall\gamma \in K^*$ that

$$\hat{f}(\gamma x(D)^{-1}) = \begin{cases} q^{1-g} & if\ v(\gamma x(D)_v^{-1}) \geqslant m_v\ \forall v\ (\Leftrightarrow \gamma \in L(\mathscr{K} - D)) \\ 0 & otherwise \end{cases}$$

We have

$$\sum_{\gamma\in K} \hat{f}(\gamma x(D)^{-1}) = q^{l(\mathscr{K}-D)-g+1}.$$

□

4 The Volume of C_K^1

For any finite set S of places of K, let us now define the S-ideles of K by

$$\mathbf{I}_{K,S} = \{x = (x_v) \in \mathbf{I}_K : |x_v| = 1, \forall v \notin S\}$$

with norm-one given by

$$\mathbf{I}_{K,S}^1 = \mathbf{I}_K^1 \cap \mathbf{I}_{K,S}.$$

Note also that $\mathbf{I}_{K,\phi} = \mathbf{I}_{K,\phi}^1$ is compact.

As in Section 1, we may extend the divisor map:

$$div : \mathbf{I}_K^1 \to Div^0(K)$$

$$(x_v) \mapsto \sum_v v(x_v)v$$

according to $\prod_v |x_v|_v = 1$ for $x = (x_v) \in \mathbf{I}_K^1$. It is easy to see that this map is surjective and $ker(div) = \mathbf{I}_{K,\phi}$. From the equality $\mathbf{I}_{K,\phi} \cap K^* = \mathbf{F}_q^*$, we get the short exact sequence:

$$1 \to \mathbf{I}_{K,\phi}/\mathbf{F}_q^* \to \mathbf{I}_K^1/\mathbf{K}^* = C_K^1 \to Pic^0(K) \to 0.$$

and

$$1 \to \mathbf{F}_q^* \to \mathbf{I}_{K,\phi} \to \mathbf{I}_{K,\phi}/\mathbf{F}_q^* \to 1.$$

And we can see that $Pic^0(K)$ is compact and discrete, and therefore finite.

Note that clearly

$$\mathbf{I}_{K,\phi} = \prod_v U_v$$

where U_v denotes the subset of elements of K_v of absolute value one. Following our normalization of Haar measure, $Vol(\mathbf{I}_{K,\phi}) = 1$.

As discussed above, we have

$$\begin{aligned} Vol(C_K^1) &= Vol(\mathbf{I}_{K,\phi}/\mathbf{F}_q^*) \cdot Vol(Pic^0(K)) \\ &= \frac{Vol(\mathbf{I}_{K,\phi})}{Vol(\mathbf{F}_q^*)} \cdot Card(Pic^0(K)) \\ &= \frac{1}{q-1} \cdot Card(Pic^0(K). \end{aligned}$$

5 The Dedekind zeta-function

Let $f = \otimes_v f_v \in S(\mathbf{A}_K)$ be defined by requiring that each component function f_v be the characteristic function on $\mathfrak{o}_v$. Then

$$Z(s, 1, f) = \prod_v (1 - q_v^{-s})^{-1}.$$

The left-hand side can be continued analytically as a meromorphic function over the whole s-plane; as the same is true for the right.

Definition 5.1. The meromorphic function ζ_K in the s-plane, given for $Re(s) > 1$, by the product

$$\zeta_K = \prod_v (1 - q_v^{-s})^{-1}$$

Weil had a conjecture about Dedekind zeta-function, however what we talk about here is the case for curves.

Theorem 5.2. Let K is a global field with char$K = p > 1$; let $\mathbf{F}_q$ be the largest finite subfield of K and g its genus. Then

$$\zeta_K(s) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

where P is a polynomial of degree 2g, such that

$$P(u) = q^g u^{2g} P(1/qu).$$

Moreover, $P(0) = 1$, and $P(1) = Card(Pic^0(K))$.

ζ_K and $Z(s, 1, f)$ have same poles, which are $s = 1$ and $s = 0$. Let u be q^{-s}, then $\zeta_K(s) = R(u) = P(u)/(1-u)(1-qu)$, where P is an entire function in the u-plane. Like the Riemann zeta-function, we can get R(u)$\rightarrow$ 1 as u$\rightarrow$ 0, so that P(0)=1.

As in section 3, $\hat{f}(x) = |\delta|^{1/2} f(\delta x)$, with $|\delta|^{1/2} = \prod_v |\delta_v|^{1/2} = q^{1-g}$. Thus we have

$$Z(s, 1, \hat{f}) = |\delta|^{1/2-s} Z(s, 1, f).$$

On the other hand, by global function equation, $Z(s, 1, \hat{f}) = Z(1-s, 1, f)$.So

$$\frac{P(q^{s-1})}{(1-q^{s-1})(1-q^s)} = Z(1-s, 1, f) = Z(s, 1, \hat{f})$$

$$= |\delta|^{1/2-s} Z(s, 1, f) = q^{(2-2g)(1/2-s)} \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})} \qquad (*).$$

Then we have $P(u) = q^g u^{2g} P(1/qu)$, and

$$1 = P(0) = q^g \lim_{u \to 0} u^{2g} P(1/qu)$$

P(u) has a Laurent extension as P(u) is entire, but by the above identity we have P(u) must be a polynomial which exactly has degree 2g. Moreover, compute the residue of s=0 by two sides of (*), we can get $P(1) = Card(Pic^0(K))$.

Weil conjecture also tells us the roots of the polynomial P(u) has absolute norm $q^{-1/2}$

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